Les invariants de Seiberg-Witten et la conjecture de van de Ven

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Seiberg-Witten Invariants and the Van De Ven Conjecture

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The purpose of this note is to give a short, selfcontained proof of the following result:

Theorem 1 *A complex surface which is diffeomeorphic to a rational surface is rational.*

This result has been announced by R. Friedman and Z. Qin [FQ]. Whereas their proof uses Donaldson theory and vector bundles techniques, our proof uses the new Seiberg-Witten invariants [W], and the interpretation of these invariants in terms of stable pairs [OT].

Combining the theorem above with the results of [FM], one obtains a proof of the Van de Ven conjecture [V]:

Corollary 2 *The Kodaira dimension of a complex surface is a differential invariant.*

Proof: (of the Theorem) It suffices to prove the theorem for algebraic surfaces [BPV]. Let X be an algebraic surface of non-negative Kodaira dimension, with $\pi_1(X)=\{1\}$ and $p_g(X)=0$. We may suppose that X is the blow up in k *distinct* points of its minimal model X_{min}. Denote the contraction to the minimal model by $\sigma : X \longrightarrow X_{\mathrm{min}}$, and the exceptional divisor by $E=\sum\limits_{i=1}^{k} E_i$.

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Fix an ample divisor $H_{\min}$ on $X_{\min}$, a sufficiently large integer n, and let $H_n := \sigma^*(nH_{\min}) - E$ be the associated polarization of X.

For every subset $I \subset \{1, \ldots, k\}$ we put $E_I := \sum_{i \in I} E_i$, and $L_I := 2[E_I] - [K_X]$, where K_X is a canonical divisor. Clearly $L_I = [E_I] - [E_{\bar{I}}] - \sigma^*([K_{\min}])$, where $\bar{I}$ denotes the complement of I in $\{1, \ldots, k\}$. The cohomology classes L_I are almost canonical classes in the sense of [OT]. Now choose a Kähler metric g_n on X with Kähler class $[\omega_{g_n}] = c_1(\mathcal{O}_X(H_n))$. Since $[\omega_{g_n}] \cdot L_I < 0$ for sufficiently large n, the main result of [OT] identifies the Seiberg-Witten moduli space $\mathcal{W}_X^{g_n}(L_I)$ with the union of all complete linear systems $|D|$ corresponding to effective divisors D on X with $c_1(\mathcal{O}_X(2D - K_X)) = L_I$.

Since $H^2(X, \mathbb{Z})$ has no 2-torsion, and $q(X) = 0$, there is only one such divisor, $D = E_I$. Furthermore, from $h^1(\mathcal{O}_X(E_I)|_{E_I}) = 0$, and the smoothness criterion in [OT], we obtain:

$$\mathcal{W}_X^{g_n}(L_I) = \{E_I\},$$

i.e. $\mathcal{W}_X^{g_n}(L_I)$ consists of a single smooth point. The corresponding Seiberg-Witten invariants are therefore odd: $n_{L_I}^{g_n} = \pm 1$.

Consider now the positive cone $\mathcal{K} := \{\eta \in H^2_{\mathrm{DR}}(X) |\ \eta^2 > 0\}$; using the Hodge index theorem, the fact that $K_{\min}$ is cohomologically nontrivial, and $K_{\min}^2 \geq 0$, we see that $\mathcal{K}$ splits as a disjoint union of two components $\mathcal{K}_\pm := \{\eta \in \mathcal{K} |\ \pm \eta \cdot \sigma^*(K_{\min}) > 0\}$. Clearly $[\omega_{g_n}]$ belongs to $\mathcal{K}_+$.

Let g be an *arbitrary* Riemannian metric on X, and let ω_g be a g-selfdual closed 2-form on X such that $[\omega_g] \in \mathcal{K}_+$.

For a fixed $I \subset \{1, \ldots, k\}$, we denote by $L_I^\perp \subset \mathcal{K}_+$ the wall associated with L_I, i.e. the subset of classes η with $\eta \cdot L_I = 0$.

Claim: The rays $\mathbb{R}_{>0}[\omega_g]$, $\mathbb{R}_{>0}[\omega_{g_n}]$ belong either to the same component of $\mathcal{K}_+ \setminus L_I^\perp$ or to the same component of $\mathcal{K}_+ \setminus L_{\bar{I}}^\perp$.

Indeed, since $[\omega_{g_n}] \cdot L_I < 0$ and $[\omega_{g_n}] \cdot L_{\bar{I}} < 0$, we just have to exclude that

$$[\omega_g] \cdot L_I \geq 0 \quad \text{and} \quad [\omega_g] \cdot L_{\bar{I}} \geq 0. \tag{$*$}$$

Write $[\omega_g] = \sum_{i=1}^{k} \lambda_i [E_i] + \sigma^*[\omega]$, for some class $[\omega] \in H^2_{\mathrm{DR}}(X_{\min})$; then $[\omega]^2 > \sum_{i=1}^{k} \lambda_i^2$, and $[\omega] \cdot K_{\min} > 0$, since ω_g was chosen such that its cohomology class belongs to $\mathcal{K}_+$. The inequalities $(*)$ can now be written as

$$-\sum_{i\in I}\lambda_i+\sum_{j\in\bar{I}}\lambda_j-[\omega]\cdot K_{\min}\geq 0\quad\text{and}\quad -\sum_{j\in\bar{I}}\lambda_j+\sum_{i\in I}\lambda_i-[\omega]\cdot K_{\min}\geq 0,$$

and we obtain the contradiction $[\omega]\cdot K_{\min}\leq 0$. This proves the claim.

We know already that the mod 2 Seiberg-Witten invariants $n^{g_n}_{L_I}(\text{mod } 2)$ and $n^{g_n}_{L_{\bar{I}}}(\text{mod } 2)$ are nontrivial for the special metric g_n. Since the invariants $n^{g}_{L_I}(\text{mod } 2)$ and $n^{g}_{L_{\bar{I}}}(\text{mod } 2)$ depend only on the chamber of the ray $\mathbb{R}_{>0}[\omega_g]$ with repect to the wall $L_I^{\perp}$, respectively $L_{\bar{I}}^{\perp}$ (see [W], [KM]), at least one of the invariants associated with the metric g must be non-zero, too.

But any rational surface admits a Hodge metric with positive total scalar curvature [H], and with respect to such a metric *all* Seiberg-Witten invariants are trivial [OT]. ■

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